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Cantor's and Baire's theorem in fuzzy soft 2-metric spaces

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Abstract. This article introduces and characterize fuzzy soft 2-metric spaces utilizing the concept of fuzzy soft points within a fuzzy soft universe. The framework of fuzzy soft 2-metric spaces allows us to explore properties, including the concept of convergence of sequences within fuzzy soft 2-metric spaces. Further, the study establishes the analysis of the Cantor's intersection theorem within the context of complete fuzzy soft 2-metric spaces.

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1. Introduction

In various domains of daily life, uncertainty permeates a wide array of fields such as engineering, physics, computer science, economics, social science, and medical science. Addressing uncertainties often involves the applications of mathematical tools, with two notable approaches being fuzzy set theory introduced by Zadeh [1] and the theory of soft sets initiated by Molodstov [2] which helps to solve problems in all area. Later, many authors contributed to the investigation of soft sets in topological and computational aspects [3, 4, 5, 6, 7]. These tools, especially when combined with fuzzy sets, offer solutions to problems across diverse areas.

A significant contribution to this field is the work by Maji et al. [8], who introduced various operations in soft sets and coined the concept of fuzzy soft sets. Building upon this, Beaula et al. [9] defined fuzzy soft metric spaces in terms of fuzzy soft points, investigating their properties. Sayed et al. [10] extended this study to fuzzy soft contractive mappings in fuzzy soft metric spaces, exploring fixed point theorems. Additionally, they delved into fuzzy soft G-metric spaces [11, 12].

The measurement of distances between points and sets holds paramount importance in human life, influencing various mathematical branches like geometry, functional analysis and computer science. [13] Gahler introduced the concept of 2-metric spaces in 1964, where the distance between points is determined by a metric function. This concept extends the idea of an ordinary metric with the geometric interpretation that d(x, y, z) represents the area of a triangle formed by vertices x, y and z in X. In 2022, Lee et al. [14] introduced the concept of neighborhood structures and continuities via cubic sets and they analyze the cubic sets in category theory.

The theory of 2-metric spaces has been extensively studied and developed over the years. Vildan et al. [15] made contributions by introducing and exploring soft 2-metric spaces. In this paper, the basic concepts related to fuzzy soft metric spaces and soft metric spaces are recalled in section 2. The new notion of fuzzy soft 2-metric spaces with fuzzy soft points is defined. This study includes an examination of fuzzy soft open 2-balls, fuzzy soft 2-closed balls and the definition of convergence sequences in fuzzy soft 2-metric spaces, accompanied by the proof of results in section 3. Furthermore, the paper establishes the Cantor's Intersection Theorem and Baire's Theorem in complete fuzzy soft 2-metric spaces in section 4.

2. Preliminaries

In this section, some basic definitions related to fuzzy soft sets, fuzzy soft points, fuzzy soft metric space, fuzzy soft G-metric space and soft 2-metric space are recalled. Throughout this paper, X be a universal set, E be set of parameters and $A, B \subseteq E$.

Definition 2.1 ([16])**.** Let $A \subseteq E$. Then a pair (f, A) denoted by f_A is called a *fuzzy soft set* over (X, E), where f is a mapping $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$, where

$$\mu^e_{f_A} = \begin{cases} \tilde{0} & \text{if } e \notin A \\ \text{otherwise} & \text{if } e \in A. \end{cases}$$

$(\widetilde{X, E})$ denotes the class of all fuzzy soft sets over (X, E) and is called a *fuzzy soft universe.*

Definition 2.2 ([16])**.** A fuzzy soft set f_A over (X, E) is said to be:
(i) a *null fuzzy soft set*, denoted by $\tilde{\phi}$, if $f_A(e) = \tilde{0}$ for all $e \in A$,
(ii) an *absolute fuzzy soft set*, denoted by $\tilde{F}_E$, if $f_A(e) = \tilde{1}$ for all $e \in A$.

Definition 2.3 ([17])**.** The *complement* of a fuzzy soft set f_A, denoted by f^c_A, where $f^c_A : E \to I^X$ is a mapping given by $\mu^e_{f^c_A} = 1 - \mu^e_{f_A}$ for all $e \in E$.

It is obvious that $(f^c_A)^c = f_A$.

Definition 2.4 ([17])**.** Let $f_A,\ g_B \in (\widetilde{X, E})$. Then we say that f_A *is fuzzy soft subset of* g_B, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \leq \mu^e_{g_B}$ for all $e \in A$, i.e., $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)$, for all $x \in X$ and for all $e \in A$.

Definition 2.5 ([17])**.** Let $f_A,\ g_B \in (\widetilde{X, E})$. Then the *union* of f_A and g_B, denoted by $h_C = f_A \cup g_B$, is a fuzzy soft set h_C over X defined as follows:
(i) $C = A \cup B$,
(ii) $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B}$ for all $e \in C$.

Definition 2.6 ([17])**.** Let $f_A,\ g_B \in \widetilde{(X,E)}$. Then the *intersection* of f_A and g_B, denoted by $h_C = f_A \cap g_B$, is a fuzzy soft set h_C over X defined as follows:
(i)$C = A \cap B$,
(ii) $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \wedge \mu^e_{g_B}$ for all $e \in C$

Definition 2.7 ([18])**.** A fuzzy soft set $f_A \in \widetilde{(X,E)}$ is called a *fuzzy soft point*, if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha(0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0$ for each $y \in X - \{x\}$ and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.8 ([18])**.** A fuzzy soft point x^e_α is said to *belong to* a fuzzy soft set g_A, denoted by $x^e_\alpha \in g_A$, if $\alpha \leq \mu^e_{g_A}(x)$ for all $e \in A$.

Definition 2.9 ([9])**.** Let f_A be fuzzy soft set over X. The two fuzzy soft points $f_{e_1},\ f_{e_2} \in f_A$ are said to *be equal*, if $\mu_{f_{e_1}}(x) = \mu_{f_{e_2}}(x)$ for all $x \in X$. Thus $f_{e_1} \neq f_{e_2}$ if and only if $\mu_{f_{e_1}}(x) \neq \mu_{f_{e_2}}(x)$ for some $x \in X$.

Definition 2.10 ([9])**.** The union of any collection of fuzzy soft points can be considered as a fuzzy soft set and every fuzzy soft set can be expressed as the union of all fuzzy soft points, i.e., $f_A = \{\bigcup_{f_e \in f_A} f_e : e \in E\}$.

Definition 2.11 ([9])**.** Let $f_A,\ f_B$ be two fuzzy soft sets. Then $f_A \subseteq f_B$ if and only if $f_e \in f_A$ implies $f_e \in f_B$ and thus $f_A = f_B$ if and only if $f_e \in f_A$ implies $f_e \in f_B$ and $f_e \in f_B$ implies $f_e \in f_A$.

Definition 2.12 ([19])**.** Let $\mathcal{R}$ be the set of real numbers and $B(\mathcal{R})$ be the collection of all non empty bounded subsets of $\mathcal{R}$, E be a set of parameters and $A \subseteq E$. Then a mapping $f : A \to B(\mathcal{R})$ is called a *soft real set*. If a soft real set is a singleton soft set, then it will be called a *soft real number* and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\overline{0}$ and $\overline{1}$ are the soft real numbers where $\overline{0}(e) = 0, \overline{1}(e) = 1$ for all $e \in E$ respectively.

The set of all soft real numbers is denoted by $\mathcal{R}(A)$ and the set of all non-negative soft real numbers by $\mathcal{R}(A)^*$.

The soft real number $\tilde{r}$ will denote a particular type of soft real number such that $\tilde{r}(e) = r$ for all $e \in E$.

Definition 2.13 ([20])**.** (i) A soft set (P, A) over X is said to be a *soft point*, denoted by P^x_λ, if there is exactly one $\lambda \in A$ such that $P^x_\lambda(\lambda) = \{x\}$ for some $x \in X$ and $P(\mu) = \varnothing$ for all $\mu \in A \setminus \{\lambda\}$.
(ii) A soft point P^x_λ is said to *belong to* a soft set (F, A), denoted by $P^x_\lambda \in (F, A)$, if $P^x_\lambda(\lambda) = \{x\} \subset F(\lambda)$.
(iii) Two soft points $P^x_\lambda,\ P^y_\mu$ are said to *be equal*, if $\lambda = \mu$ and $x = y$. Then $P^x_\lambda \neq P^y_\mu$ iff $x \neq y$ or $\lambda \neq \mu$.
The collection of all soft points of $\tilde{X}$ is denoted by $SP(\tilde{X})$.

Definition 2.14 ([21])**.** Let U be a universe set, let A be a set of parameters and let $SS(U)_A$ be the family of all soft sets over U with respect to A. Then $(F, A) \in SS(U)_A$ is called a *soft point* in $\tilde{U}_A$, denoted by e_F, if $e_F(e) \neq \phi$ and $F(e^{'}) = \phi$ for all $e^{'} \in A - \{e\}$.

Definition 2.15 ([4])**.** Let U be a universe and A a parameter set. Then $\tau \subseteq SS(U)_A$ is called a *soft topology* on U, if it satisfies the following conditions:

(T1) $\tilde{\phi}$, $\tilde{U}_A \in \tau$,
(T2) the union of any number of soft sets in τ belongs to τ,
(T3) the intersection of any two soft sets in τ belongs to τ.
The triplet (U, τ, A) is called a *soft topological space* over U. The members of τ are called *soft open sets in* U and their complements are called *soft closed sets in* U.

Definition 2.16 ([21]). A soft set (G, A) in a soft topological space (U, τ, A) is called a soft neighborhood of the soft point $e_F \in \tilde{U}_A$, if there exists $(H, A) \in \tau$ such that $e_F \in (H, A) \subseteq (G, A)$.

Definition 2.17 ([10]). A *fuzzy soft real number* (*non-negative*) is a fuzzy set of all (non-negative) soft real numbers $\mathcal{R}(A)$, i.e., a mapping $\tilde{\tilde{\lambda}} : \mathcal{R}(A) \to [0, 1]$, associating with each soft real number $\tilde{t}$, its grade of membership $\tilde{\tilde{\lambda}}(\tilde{t})$ satisfying the following conditions:
(i) $\tilde{\tilde{\lambda}}$ is convex, i.e., $\tilde{\tilde{\lambda}}(\tilde{t}) \geq \min(\tilde{\tilde{\lambda}}(\tilde{s}), \tilde{\tilde{\lambda}}(\tilde{r}))$ for $\tilde{s} \subseteq \tilde{t} \subseteq \tilde{r}$,
(ii) $\tilde{\tilde{\lambda}}$ is normal, i.e., there exists $\tilde{t}_0 \in \mathcal{R}(A)^*$ such that $\tilde{\tilde{\lambda}}(\tilde{t}_0) = 1$,
(iii) $\tilde{\tilde{\lambda}}$ is upper semi continuous provided for all $\tilde{t} \in \mathcal{R}(A)$ and $\alpha \in [0, 1]$ $\tilde{\tilde{\lambda}}(\tilde{t}) < \alpha$, there is a $\delta > 0$ such that $||\tilde{s} - \tilde{t}|| \leq \delta$ implies that $\tilde{\tilde{\lambda}}(\tilde{s}) < \alpha$.

The fuzzy soft real numbers be denoted by $\tilde{\tilde{r}}, \tilde{\tilde{s}}, \tilde{\tilde{t}}$ etc., while $\tilde{\tilde{r}}, \tilde{\tilde{s}}, \tilde{\tilde{t}}$ will be denoted in particular type of fuzzy soft real numbers such that $\tilde{\tilde{r}}(e) = \mu^e\tilde{r}$ that is a fuzzy number for all $e \in E$.

Let $A \subseteq E$. Then the set of all non negative fuzzy soft real numbers is denoted by $\mathbb{R}(A)^*$ and the collection of all fuzzy soft points of a fuzzy soft set f_A over X be denoted by $FSC(f_A)$.

Definition 2.18 ([9]). Let E be a set of parameters, $A \subseteq E$ and $\tilde{F}_E$ be a absolute fuzzy soft set. Then a mapping $d : FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \to \mathbb{R}(A)^*$ is said to be a *fuzzy soft metric* on $\tilde{F}_E$, if d satisfies the following conditions:
(FSM$_1$) $\tilde{d}(f_{e_1}, f_{e_2}) \geq \tilde{0}$ for all $f_{e_1}, f_{e_2} \in \tilde{F}_E$,
(FSM$_2$) $\tilde{d}(f_{e_1}, f_{e_2}) = \tilde{0}$ if and only if $f_{e_1} = f_{e_2}$ for all $f_{e_1}, f_{e_2} \in \tilde{F}_E$,
(FSM$_3$) $\tilde{d}(f_{e_1}, f_{e_2}) = \tilde{d}(f_{e_2}, f_{e_1})$ for all $f_{e_1}, f_{e_2} \in \tilde{F}_E$,
(FSM$_4$) $\tilde{d}(f_{e_1}, f_{e_3}) \leq \tilde{d}(f_{e_1}, f_{e_2}) + \tilde{d}(f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3} \in \tilde{F}_E$.
The fuzzy soft set $\tilde{F}_E$ with a fuzzy soft metric $\tilde{d}$ is called a *fuzzy soft metric space* and is denoted by $(\tilde{F}_E, d)$.

Definition 2.19 ([11]). Let E be the set of parameters, $A \subseteq E$ and $\tilde{F}_E$ be a absolute fuzzy soft set. Then a mapping $\tilde{G} : FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \to \mathbb{R}(A)^*$ is said to be a *fuzzy soft G-metric* on $\tilde{F}_E$, if $\tilde{G}_E$ satisfies the following conditions:
(FS$\tilde{G}_1$) $\tilde{G}(f_{e_1}, f_{e_2}, f_{e_3}) = \tilde{0}$, if $f_{e_1} = f_{e_2} = f_{e_3}$,
(FS$\tilde{G}_2$) $\tilde{0} \tilde{<} \tilde{G}(f_{e_1}, f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2} \tilde{\in} FSC(\tilde{E})$ with $f_{e_1} \neq f_{e_2}$,
(FS$\tilde{G}_3$) $\tilde{G}(f_{e_1}, f_{e_1}, f_{e_2}) \leq \tilde{G}(f_{e_1}, f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3} \tilde{\in} FSC(\tilde{E})$ with $f_{e_2} \neq f_{e_3}$,
(FS$\tilde{G}_4$) $\tilde{G}(f_{e_1}, f_{e_2}, f_{e_3}) = \tilde{G}(f_{e_1}, f_{e_3}, f_{e_2}) = \tilde{G}(f_{e_2}, f_{e_3}, f_{e_1}) = \cdots$,
(FS$\tilde{G}_5$) $\tilde{G}(f_{e_1}, f_{e_2}, f_{e_3}) \leq \tilde{G}(f_{e_1}, f_e, f_e) + \tilde{G}(f_e, f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3}, f_e \in FSC(\tilde{F}_E)$.

The fuzzy soft set $\tilde{F}_E$ with a fuzzy soft G-metric $\tilde{G}$ on $\tilde{F}_E$ is called a *fuzzy soft G-metric space* and is denoted by $(\tilde{F}_E, \tilde{G})$.

Definition 2.20 ([15]). A mapping $d : SP(\tilde{X}) \times SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(A)^*$ is said to be a *soft 2-metric* on the soft set $\tilde{X}$, if d satisfies the following conditions:

(M1) for any $P^x_\lambda,\ P^y_\mu \in SP(\tilde{X})$ with $P^x_\lambda \neq P^y_\mu$, there exists $P^z_\gamma \in SP(\tilde{X})$ such that $d(P^x_\lambda, P^y_\mu, P^z_\gamma) \neq \tilde{0}$,

(M2) $d(P^x_\lambda, P^y_\mu, P^z_\gamma) = \tilde{0}$, when at least two of three soft points are same, i.e., $P^x_\lambda = P^y_\mu$ or $P^y_\mu = P^z_\gamma$ or $P^x_\lambda = P^y_\mu = P^z_\gamma$

(M3) $d(P^x_\lambda, P^y_\mu, P^z_\gamma) = d(P^y_\mu, P^x_\lambda, P^z_\gamma) = d(P^z_\gamma, P^y_\mu, P^x_\lambda)$ for all $P^x_\lambda, P^y_\mu, P^z_\gamma \in SP(\tilde{X})$,

(M4) $d(P^x_\lambda, P^y_\mu, P^z_\gamma) \tilde{\leq} d(P^y_\mu, P^x_\lambda, P^t_\eta) + d(P^y_\mu, P^t_\eta, P^z_\gamma) + d(P^t_\eta, P^y_\mu, P^z_\gamma)$ for all $P^x_\lambda, P^y_\mu, P^z_\gamma$, $P^t_\eta \in SP\ (\tilde{X})$.

The soft set $\tilde{X}$ with a soft 2-metric d on $\tilde{X}$ is called a *soft 2-metric space* and denoted by the $(\tilde{X}, d)$.

3. Fuzzy soft 2-metric spaces

In this section, the new notion of fuzzy soft 2-metric space is defined, along with fundamental definitions of fuzzy soft open (closed) 2-ball, convergence of a sequence of fuzzy soft point, completeness, continuity of a fuzzy soft function and studied their properties.

Definition 3.1. Let E be a set of parameters, $A \subseteq E$ and $\tilde{F}_E$ be a absolute fuzzy soft set. Then a mapping $d : FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \to \mathbb{R}(A)^*$ is said to be a *fuzzy soft 2-metric* on $\tilde{F}_E$, if d satisfies the following conditions:

(FSM1) for any $f_{e_1},\ f_{e_2} \in FSC(\tilde{F}_E)$ with $f_{e_1} \neq f_{e_2}$, there exists $f_{e_3} \in FSC(\tilde{F}_E)$ such that $d(f_{e_1}, f_{e_2}, f_{e_3}) \neq \tilde{0}$,

(FSM2) $d(f_{e_1}, f_{e_2}, f_{e_3}) = \tilde{0}$, when at least two of three of $f_{e_1}, f_{e_2}, f_{e_3}$ are equal, i.e., $f_{e_1} = f_{e_2}$ or $f_{e_2} = f_{e_3}$ or $f_{e_1} = f_{e_2} = f_{e_3}$,

(FSM3) $d(f_{e_1}, f_{e_2}, f_{e_3}) = d(f_{e_2}, f_{e_1}, f_{e_3}) = d(f_{e_3}, f_{e_2}, f_{e_1})$ for all $f_{e_1}, f_{e_2}, f_{e_3} \in FSC(\tilde{F}_E)$,

(FSM4) $d(f_{e_1}, f_{e_2}, f_{e_3}) \leq d(f_{e_1}, f_{e_2}, f_e) + d(f_{e_1}, f_e, f_{e_3}) + d(f_e, f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3}, f_e \in FSC\ (\tilde{F}_E)$.

The fuzzy soft set $\tilde{F}_E$ with a fuzzy soft 2-metric d on $\tilde{F}_E$ is called a *fuzzy soft 2-metric space* and denoted by the $(\tilde{F}_E, d)$.

Example 3.2. Let $\tilde{F}_E$ be an absolute fuzzy soft set. Define $d : FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \times FSC(\tilde{F}_E) \to \mathbb{R}(A)^*$ by:

$$d(f_{e_1}, f_{e_2}, f_{e_3}) = \begin{cases} \tilde{0} & \text{if } f_{e_1} = f_{e_2} = f_{e_3} \text{ or at least two of } f_{e_1}, f_{e_2}, f_{e_3} \text{ are equal,} \\ \tilde{1} & \text{otherwise.} \end{cases}$$

Then d satisfies all the conditions of the definition 3.1. Thus d is a fuzzy soft 2-metric on the fuzzy soft set $\tilde{F}_E$.

Definition 3.3. A fuzzy soft 2-metric space $(\tilde{F}_E, d)$ is said to be *bounded*, if there exists non negative fuzzy soft real number $\tilde{\tilde{k}}$ such that $d(f_{e_1}, f_{e_2}, f_{e_3}) \leq \tilde{\tilde{k}}$, for all $f_{e_1}, f_{e_2}, f_{e_3} \in FSC(\tilde{F}_E)$. Otherwise, then $(\tilde{F}_E, d)$ is said to be *unbounded*.

Definition 3.4. Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space and let $f_{e_1}, f_{e_2} \in FSC(\tilde{F}_E)$, and $\tilde{\tilde{r}} \in \mathbb{R}(A)^*$. Then A *fuzzy soft open 2-ball centered at* f_{e_1} *and* f_{e_2} *with radius* $\tilde{\tilde{r}}$, denoted by $B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_2})$, is the collection of all fuzzy soft points f_{e_3} of $\tilde{F}_E$ such that $d(f_{e_1}, f_{e_2}, f_{e_3}) < \tilde{\tilde{r}}$, i.e.,

$$B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_2}) = \left\{ f_{e_3} \in FSC(\tilde{F}_E) \;\middle|\; d(f_{e_1}, f_{e_2}, f_{e_3}) < \tilde{\tilde{r}} \right\}.$$

A *fuzzy soft closed 2-ball*, denoted by $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$, is defined as follows:

$$B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}] = \left\{ f_{e_3} \in FSC(\tilde{F}_E) \;\middle|\; d(f_{e_1}, f_{e_2}, f_{e_3}) \le \tilde{\tilde{r}} \right\}.$$

Example 3.5. Consider the fuzzy soft 2-metric space (F_E, d) defined in Example 3.2. Then

$$B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_2}) = \begin{cases} FSC(\tilde{F}_E) & \text{if } \tilde{\tilde{r}} > \tilde{1} \\ \{f_{e_1}, f_{e_2}\} & \text{if } \tilde{\tilde{r}} \le \tilde{1}, \end{cases}$$

$$B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}] = \begin{cases} FSC(\tilde{F}_E) & \text{if } \tilde{\tilde{r}} \ge \tilde{1} \\ \{f_{e_1}, f_{e_2}\} & \text{if } \tilde{\tilde{r}} < \tilde{1}. \end{cases}$$

Definition 3.6. Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space. Then $(\tilde{F}_E, d)$ is called a T_1*-space*, if for any $f_{e_1},\ f_{e_2} \in \tilde{F}_E$ with $f_{e_1} \neq f_{e_2}$, there are two fuzzy soft 2-open balls $B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_3})$ and $B_{\tilde{\tilde{r}}}(f_{e_2}, f_{e_3})$ with centers f_{e_1}, f_{e_3} and f_{e_2}, f_{e_3} respectively and radius $\tilde{\tilde{r}} > \tilde{0}$ such that $f_{e_1} \in B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_3}), f_{e_2} \notin B_{\tilde{\tilde{r}}}(f_{e_1}, f_{e_3})$ and $f_{e_1} \notin B_{\tilde{\tilde{r}}}(f_{e_2}, f_{e_3}),\ f_{e_2} \in B_{\tilde{\tilde{r}}}(f_{e_2}, f_{e_3})$.

Theorem 3.7. *Every fuzzy soft 2-metric space is a* T_1*-space.*

Proof. Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space and let $f_{e_1},\ f_{e_2} \in FSC(\tilde{F}_E)$ with $f_{e_1} \neq f_{e_2}$. Then there exists $f_{e_3} \in FSC(\tilde{F}_E)$ such that $d(f_{e_1}, f_{e_2}, f_{e_3}) = \tilde{\tilde{r}} > \tilde{0}$. If $\tilde{\tilde{t}} = \frac{\tilde{\tilde{r}}}{2}$, then $B_{\tilde{\tilde{t}}}(f_{e_1}, f_{e_3})$ and $B_{\tilde{\tilde{t}}}(f_{e_2}, f_{e_3})$ are two fuzzy soft open 2-balls with $f_{e_1} \in B_{\tilde{\tilde{t}}}(f_{e_1}, f_{e_3}),\ f_{e_2} \notin B_{\tilde{\tilde{t}}}(f_{e_1}, f_{e_3})$ but $f_{e_2} \in B_{\tilde{\tilde{t}}}(f_{e_2}, f_{e_3}),\ f_{e_1} \notin B_{\tilde{\tilde{t}}}(f_{e_2}, f_{e_3})$. ☐

Remark 3.8. The collection of all fuzzy soft open 2-balls, taken as a subbasis, allows us to define a fuzzy soft topology on $\tilde{F}_E$, which we will refer to as the fuzzy soft 2-metric topology and denote this topology as $\tilde{\tau}_d$. Then $(\tilde{F}_E, \tilde{\tau}_d)$ is a fuzzy soft 2-metric topological space. Members of $\tilde{\tau}_d$ are called *fuzzy soft 2-open sets* and their complements are called *fuzzy soft 2-closed sets*.

Lemma 3.9. *A fuzzy soft subset* $\tilde{G}_E$ *of* $(\tilde{F}_E, \tilde{\tau}_d)$ *is fuzzy soft 2-open if and only if for any* $f_e \in \tilde{G}_E$*, there are finite number of fuzzy soft points* $f_{e_1}, f_{e_2}, f_{e_3}, \cdots, f_{e_n} \in FSC(\tilde{F}_E), \tilde{\tilde{r}}_1, \tilde{\tilde{r}}_2, \tilde{\tilde{r}}_3, \cdots, \tilde{\tilde{r}}_n > \tilde{0}$ *such that* $f_e \in B_{\tilde{\tilde{r}}_1}(f_e, f_{e_1}) \cap B_{\tilde{\tilde{r}}_2}(f_e, f_{e_2}) \cap B_{\tilde{\tilde{r}}_3}(f_e, f_{e_3}) \cap \cdots \cap B_{\tilde{\tilde{r}}_n}(f_e, f_{e_n}) \subset \tilde{G}_E$.

Proof. Suppose the necessary condition holds. Since each $B_{\tilde{\tilde{r}}_i}(f_e, f_{e_i}), i = 1, 2, \cdots, n$, is a fuzzy soft 2-open set by definition, the sufficiency of the condition follows immediately.

Conversely, suppose $\tilde{G}_E$ is a fuzzy soft 2-open set and let $f_e \in \tilde{G}_E$. Then there exist finite number of fuzzy soft open 2-balls $B_{\tilde{\tilde{r}}_i}(f_{e_i'}, f_{e_i}), i = 1, 2, 3, \cdots, m$, such that

$$f_e \in \bigcap_{i=1}^{m} B_{\tilde{\tilde{r}}_i}(f_{e_i'}, f_{e_i}) \subset \tilde{G}_E.$$

Since $f_e \in B_{\tilde{\tilde{r}}_i}(f_{e_i'}, f_{e_i})$, $d(f_e, f_{e_i'}, f_{e_i}) = \tilde{\tilde{s}}_i < \tilde{\tilde{r}}_i$. Choose $\tilde{\tilde{t}}_i < \frac{\tilde{\tilde{r}}_i - \tilde{\tilde{s}}_i}{2}$. Then $B_{\tilde{\tilde{t}}_i}(f_e, f_{e_i}) \cap B_{\tilde{\tilde{t}}_i}(f_e, f_{e_i'}) \subset B_{\tilde{\tilde{r}}_i}(f_{e_i'}, f_{e_i})$ and this true for $i = 1, 2, 3, \cdots, m$. Thus we have

$$\begin{aligned} f_e &\in B_{\tilde{\tilde{t}}_1}(f_e, f_{e_i}) \cap B_{\tilde{\tilde{t}}_1}(f_e, f_{e_i'}) \cap \cdots \cap B_{\tilde{\tilde{t}}_m}(f_e, f_{e_m}) \cap B_{\tilde{\tilde{t}}_m}(f_e, f_{e_m'}) \\ &\subset \textstyle\bigcap_{i=1}^{m} B_{\tilde{\tilde{r}}_i}(f_{e_i'}, f_{e_i}) \\ &\subset \tilde{G}_E. \end{aligned}$$

□

Definition 3.10. Let $(\tilde{F}_E, \tilde{\tau}_d)$ is a fuzzy soft 2-metric topological space and $\tilde{G}_E \subset \tilde{F}_E$. Then f_e is called a *fuzzy soft 2-limit point* of $\tilde{G}_E$, if for any fuzzy soft 2-open set $\tilde{H}_E$ containing f_e, $\tilde{H}_E \cap (\tilde{G}_E \setminus f_e) \neq \phi$.
The set of all fuzzy soft 2-limit points of $\tilde{G}_E$ is denoted by $\tilde{G}_E^{'}$.

Definition 3.11. Let $(\tilde{F}_E, \tilde{\tau}_d)$ is a fuzzy soft 2-metric topological space and $\tilde{G}_E \subset \tilde{F}_E$. Then the collection of all fuzzy soft points of $\tilde{G}_E$ and all fuzzy soft 2-limit points of $\tilde{G}_E$, i.e., $\tilde{G}_E \cup \tilde{G}_E^{'}$ is called the *set of fuzzy soft 2-closure points* of $\tilde{G}_E$ in $(\tilde{F}_E, \tilde{\tau}_d)$ and it is denoted by $\overline{\tilde{G}_E}$. The *fuzzy soft 2-interior* of $\tilde{G}_E$, denoted by $\tilde{G}_E^{\circ}$, is defined to be the union of all fuzzy soft 2- open sets contained in $\tilde{G}_E$.

Theorem 3.12. *Let $(\tilde{F}_E, \tilde{\tau}_d)$ is a fuzzy soft 2-metric topological space and $\tilde{G}_E \subset \tilde{F}_E$. Then $\tilde{G}_E$ is fuzzy soft 2-closed if and only if $\overline{\tilde{G}_E} = \tilde{G}_E$.*

Theorem 3.13. *In a fuzzy soft 2-metric topological space, every closed fuzzy soft 2-ball is a fuzzy soft 2-closed.*

Proof. Let $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$ be a fuzzy soft closed 2-ball centered at f_{e_1}, f_{e_2} with radius $\tilde{\tilde{r}}$ in a fuzzy soft 2-metric topological space $(\tilde{F}_E, \tilde{\tau}_d)$. Now we will show that no fuzzy soft point outside $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$ is a fuzzy soft 2-limit point of $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$. Let f_{e_3} be a fuzzy soft 2-limit point of $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$ such that $f_{e_3} \notin B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$. Then $d(f_{e_1}, f_{e_2}, f_{e_3}) > \tilde{\tilde{r}}$. Let $\tilde{\tilde{\varepsilon}} > \tilde{0}$. Since $B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_{e_3}) \cap B_{\tilde{\tilde{\varepsilon}}}(f_{e_2}, f_{e_3})$ is a fuzzy soft 2-open set containing f_{e_3}, there exists $f_e \in B_{\tilde{\tilde{\varepsilon}}}[f_{e_1}, f_{e_2}] \cap \{B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_{e_3}) \cap B_{\tilde{\tilde{\varepsilon}}}(f_{e_2}, f_{e_3}) \setminus f_{e_3}\}$. Thus the following inequality holds:

$$d(f_{e_1}, f_{e_2}, f_{e_3}) \leq d(f_{e_1}, f_{e_2}, f_e) + d(f_{e_1}, f_e, f_{e_3}) + d(f_e, f_{e_2}, f_{e_3}) < \tilde{\tilde{r}} + \tilde{\tilde{\varepsilon}} + \tilde{\tilde{\varepsilon}} = 2\tilde{\tilde{\varepsilon}} + \tilde{\tilde{r}}$$

Since $\tilde{\tilde{\varepsilon}} > 0$ is arbitrary, this implies $d(f_{e_1}, f_{e_2}, f_{e_3}) \leq \tilde{\tilde{r}}$ which contradicts the assumption. So $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$ contains all its fuzzy soft 2-limit points. Hence $B_{\tilde{\tilde{r}}}[f_{e_1}, f_{e_2}]$ is fuzzy soft 2-closed. The proof is complete. □

Definition 3.14. Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space and $\{f_{e_n}\}$ be a sequence of fuzzy soft points in $\tilde{F}_E$. Then $\{f_{e_n}\}$ is said to *converge to* $f_{e'} \in \tilde{F}_E$, if for any $f_e \in \tilde{F}_E$, $d(f_{e_n}, f_{e'}, f_e) \to \tilde{0}$ as $n \to \infty$, i.e., for every $\tilde{\tilde{\varepsilon}} > \tilde{0}$, there exists positive integer $N = N(\tilde{\tilde{\varepsilon}})$ such that $d(f_{e_n}, f_{e'}, f_e) < \tilde{\tilde{\varepsilon}}$ for all $f_e \in \tilde{F}_E$, whenever $n > N$, i.e., $n > N$ implies $f_{e_n} \in B_{\tilde{\tilde{r}}}(f_{e'}, f_e)$ for all $f_e \in \tilde{F}_E$.
It is denoted as $\{f_{e_n}\} \to f_{e'}$ as $n \to \infty$ or $lim_{n\to\infty} f_{e_n} = f_{e'}$.

Theorem 3.15. *In a fuzzy soft 2-metric space, a sequence of fuzzy soft points converges at most one fuzzy soft point of the space.*

Lemma 3.16. *A sequence $\{f_{e_n}\}$ of fuzzy soft points is convergent to f_e in $(\tilde{F}_E, d)$ if and only if for any fuzzy soft 2-open set $\tilde{G}_E$ containing f_e there exits positive integer m such that $f_{e_n} \in \tilde{G}_E$ for all $n \geq m$.*

Proof. Suppose the necessary condition holds and let $f_{e'} \in \tilde{F}_E$ and $\tilde{\tilde{\varepsilon}} > 0$. Since $B_{\tilde{\tilde{\varepsilon}}}(f_e, f_{e'})$ is a fuzzy soft 2-open set containing $f_{e'}$, there exists $m \in N$ such that $f_{e_n} \in B_{\tilde{\tilde{\varepsilon}}}(f_e, f_{e'}) \forall n \geq m$, i.e., $d(f_{e_n}, f_{e'}, f_e) < \tilde{\tilde{\varepsilon}}\ \forall\ n \geq m$. Then $d(f_{e_n}, f_{e'}, f_e) \to \tilde{0}$ as $n \to \infty$. Thus $\{f_{e_n}\}$ converges to $f_{e'}$.

Conversely, suppose $\{f_{e_n}\}$ of fuzzy soft points is convergent to f_e in $(\tilde{F}_E, d)$ and let $\tilde{G}_E$ be a fuzzy soft 2-open set with $f_e \in \tilde{G}_E$. Then from Lemma 3.9, we have

$$f_e \in B_{\tilde{\tilde{r_1}}}(f_e, f_{e_1}) \cap B_{\tilde{\tilde{r_2}}}(f_e, f_{e_2}) \cap B_{\tilde{\tilde{r_3}}}(f_e, f_{e_3}) \cap ... \cap B_{\tilde{\tilde{r_n}}}(f_e, f_{e_n}) \subset \tilde{G}_E$$

for some $f_{e_1}, f_{e_2}, f_{e_3}, \cdots, f_{e_n} \in FSC(\tilde{F_E})$ and $r_1, r_2, \cdots, r_n > \tilde{0}$.
Since $d(f_{e_n}, f_{e'}, f_e) \to \tilde{0}$ as $n \to \infty$, there exists $m_i \in N$ such that $d(f_{e_n}, f_{e'}, f_{e_i}) < \tilde{r}_i$ for all $n \geq m_i$, i.e.,

$$f_{e_n} \in B_{\tilde{\tilde{r_1}}}(f_e, f_{e_1}) \cap B_{\tilde{\tilde{r_2}}}(f_e, f_{e_2}) \cap B_{\tilde{\tilde{r_3}}}(f_e, f_{e_3}) \cap \cdots \cap B_{\tilde{\tilde{r_n}}}(f_e, f_{e_n}) \subset \tilde{G}_E \text{ for all } n \geq m.$$

Thus the necessary condition holds. □

Theorem 3.17. *Let $(\tilde{F}_E, \tau_d)$ be a fuzzy soft 2-metric topological space, $\tilde{G}_E \subset \tilde{F}_E$ and $f_{e_1} \in \tilde{F}_E$. If a sequence $\{f_{e_n}\}$ of fuzzy soft points of $\tilde{F}_E$ other than f_{e_1} converges to f_{e_1}, then $f_{e_1} \in \tilde{F}_E$ is a fuzzy soft 2-limit point of $\tilde{F}_E$.*

Proof. By definition, the proof follows. □

Definition 3.18. A sequence $\{f_{e_n}\}$ of fuzzy soft points in $(\tilde{F}_E, d)$ is said to be *fuzzy soft 2-bounded*, if for all $f_e \in \tilde{F}_E$, there exists fuzzy soft real number $M > \tilde{0}$ such that $d(f_{e_n}, f_{e_m}, f_e) \leq M$ for all $f_e \in \tilde{F}_E$ and all $m,\ n \in N$.

Definition 3.19. Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space and $\{f_{e_n}\}$ be a sequence of fuzzy soft points in $(\tilde{F}_E, d)$. Then $\{f_{e_n}\}$ is said to be a *Cauchy sequence* in $(\tilde{F}_E, d)$, if for each $\epsilon > 0$, there exists $m \in N$ such that $d(f_{e_i}, f_{e_j}, f_e) < \tilde{\tilde{\varepsilon}}$ for all $f_e \in \tilde{F}_E$ and for all $i,\ j \geq m$, i.e., $d(f_{e_i}, f_{e_j}, f_e) \to 0$ as $i, j \to \infty$.

Definition 3.20. A fuzzy soft 2-metric space $(\tilde{F}_E, d)$ is said to be *complete*, if every Cauchy sequence in $\tilde{F}_E$ converges to some fuzzy soft point of $\tilde{F}_E$.

Definition 3.21. A fuzzy soft subset $\tilde{G}_E \subset \tilde{F}_E$ is said to be *dense*, if $\overline{\tilde{G}_E} = \tilde{F}_E$.

Definition 3.22. A fuzzy soft subset $\tilde{G}_E \subset \tilde{F}_E$ is said to be *no-where dense*, if $(\overline{\tilde{G}_E})^0 = \phi$.

Definition 3.23. Let $(\tilde{F}_E, d)$ and $(\tilde{F}'_E, \rho)$ be two fuzzy soft 2-metric spaces and $\varphi_\psi : (\tilde{F}_E, d) \to (\tilde{F}'_E, \rho)$ be a fuzzy soft mapping between induced fuzzy soft 2-metric topological spaces. Then φ_ψ is said to be *continuous at* $f_e \in FSP(\tilde{F}_E)$, if for any fuzzy soft 2-open set $\tilde{U}_E$ containing $\varphi_\psi(f_e)$ in $\tilde{F}'_E$, there exists fuzzy soft 2-open set $\tilde{V}_E$ containing f_e in $\tilde{F}_E$ such that $\varphi_\psi(\tilde{U}_E) \subseteq \tilde{V}_E$. If the fuzzy soft mapping φ_ψ is continuous at every fuzzy soft points of $\tilde{F}_E$, then φ_ψ is said to be *continuous on* $\tilde{F}_E$.

Definition 3.24. A map $\varphi_\psi : (\tilde{F}_E, d) \to (\tilde{F}_E^{'}, \rho)$ is said to be *sequentially continuous*, if for each sequence of fuzzy soft points which satisfies $f_{e_n} \to f_e$ in $\tilde{F}_E$ implies that $\varphi_\psi(f_{e_n}) \to \varphi_\psi(f_e)$ in $\tilde{F}_E^{'}$.

Lemma 3.25. *Every continuous fuzzy soft mapping between fuzzy soft 2-metric topological spaces is also sequentially continuous.*

4. Cantor's and Baire's theorem in fuzzy soft 2-metric spaces

In this section, it establish an analog of Cantor's intersection theorem for complete fuzzy soft 2-metric spaces use it to show that such a space cannot be expressed as a countable union of no-where dense fuzzy soft sets.

Let $\tilde{G}_E \subset \tilde{F}_E$ and $f_e \in FSC(\tilde{F}_E)$, $\delta_{f_e}(\tilde{G}_E) = sup\{d(f_{e_1}, f_{e_2}, f_e) : f_{e_1}, f_{e_2} \in FSC(\tilde{F_E})\}$.

The quantity $\delta_{f_e}(\tilde{G}_E)$ need not be consider as the diameter of $\tilde{G}_E$. However, if $(\tilde{F}_E, d)$ is fuzzy soft 2-bounded in the sense of Definition 3.3, then for every $\tilde{G}_E \subset (\tilde{F}_E), \delta_{f_e}(\tilde{G}_E)$ is finite.

Theorem 4.1. *Let $(\tilde{F}_E, d)$ is a complete fuzzy soft 2-metric space. If $\{\tilde{F}_{e_n}\}$ is a sequence of fuzzy soft 2-closed sets with $\tilde{F}_{e_1} \supset \tilde{F}_{e_2} \cdots \supset \tilde{F}_{e_n} \supset \cdots$ such that $\delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0}$ as $n \to \infty$ $\forall$ $f_e \in \tilde{F}_E$, then $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n}$ is non-empty and contains at most one point.*

Proof. Suppose the sufficient condition holds, and let $(\tilde{F}_E, d)$ be a complete fuzzy soft 2-metric space and let $\{f_{e_n}\}$ be a sequence of fuzzy soft points of $\tilde{F}_{e_n}$. We show that $\{f_{e_n}\}$ is a Cauchy sequence in $(\tilde{F}_E, d)$. Since $\{\tilde{F}_{e_n}\}$ is decreasing, $f_{e_m} \in F_{e_n}$ $\forall$ $m \geq n$. Then for any $f_e \in FSC(\tilde{F}_E)$ and $m \geq n$, we have

$$d(f_{e_m}, f_{e_n}, f_e) \leq \delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0} \text{ as } m,\ n \to \infty..$$

Thus $\{f_{e_n}\}$ is a Cauchy sequence in $(\tilde{F}_E, d)$. Since $(\tilde{F}_E, d)$ is complete, $f_{e_n} \to f_e$.

Now we prove that $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n} \neq \varnothing$. Suppose $f_{e_k} \neq f_e$ for some k onwards, otherwise there is nothing to prove. Let $n \in N$ be fixed. Let $\tilde{U}_E$ be any fuzzy soft 2-open set containing f_e. Then by Lemma 3.12, there is $n_1 \in N$ such that $f_{e_k} \in \tilde{U}_E$ $\forall$ $k \geq n_1$. Thus $f_{e_k} \in (\tilde{U}_E \setminus f_e) \cap \tilde{F_{e_n}}$ $\forall$ $k \geq max\{n, n_1\}$. This shows that $f_e \in \overline{\tilde{F_{e_n}}} = \tilde{F}_{e_n}$, since $\tilde{F}_{e_n}$ is fuzzy soft 2-closed. As this is true for all $n \in N, f_e \in \bigcap_{n=1}^{\infty} \tilde{F_{e_n}}$. So $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n} \neq \varnothing$.

Finally, we prove that $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n}$ contains at most one point. If possible, let us assume that it contains two distinct points $f_e \neq f_{e_1}$. Choose $f_{e'} \neq f_e, f_{e_1}$. From the definition of $d(f_e, f_{e_1}, f_{e'}) \leq \delta_{f_{e'}}(\tilde{F}_{e_n})$, since $\delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0}$ *as* $n \to \infty$ $d(f_e, f_{e_1}, f_{e'}) = \tilde{0}$, which is a contradiction. Then $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n}$ contains at most one point. This completes the proof. □

To prove the converse of Theorem 4.1, establish the following lemma.

Lemma 4.2. *Let $\tilde{G}_E \subset \tilde{F}_E$ and $f_e \in FSC(\tilde{F}_E)$. Then $\delta_{f_e}(\tilde{G}_E) = \delta_{f_e}(\overline{\tilde{G}_E})$.*

Proof. Since $\tilde{G}_E \subset \overline{\tilde{G}}_E$, it follows that $\delta_{f_e}(\tilde{G}_E) \leq \delta_{f_e}(\overline{\tilde{G}}_E)$. To prove the converse, let $f_{e_1},\ f_{e_2} \in \overline{\tilde{G}}_E$. If $f_{e_1},\ f_{e_2} \in FSC(\tilde{G}_E)$, then clearly $d(f_{e_1}, f_{e_2}, f_e) \leq \delta_{f_e}(\tilde{G}_E)$. Thus suppose first that one of them, say, $f_{e_1} \notin FSC(\tilde{G}_E)$ and $f_{e_2} \in FSC(\tilde{G}_E)$. Let $\tilde{\tilde{\varepsilon}} > \tilde{0}$ be arbitrary. Since $f_{e_1} \in \overline{\tilde{G}}_E$ and $B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_{e_2}) \cap B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_e)$ is a fuzzy soft 2-open set containing f_{e_1}, there exists $f_{e_3} \in \tilde{G}_E[B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_{e_2}) \cap B_{\tilde{\tilde{\varepsilon}}}(f_{e_1}, f_e)]$. Then

$$\begin{aligned} d(f_{e_1}, f_{e_2}, f_e) &\leq d(f_{e_1}, f_{e_2}, f_{e_3}) + d(f_{e_1}, f_{e_3}, f_{e_2}) + d(f_{e_3}, f_{e_2}, f_{e_1}) \\ &< 2\tilde{\tilde{\varepsilon}} + \delta_{f_e}(\tilde{G}_E). \end{aligned}$$

Since, this is true for every $\tilde{\tilde{\varepsilon}} > \tilde{0}$, conclude that $d(f_{e_1}, f_{e_2}, f_e) \leq \delta_{f_e}(\tilde{G}_E)$ for $f_{e_2} \in FSC(\tilde{G}_E)$ and $f_{e_1} \in \overline{\tilde{G}}_E$. Finally, if $f_{e_1}, f_{e_2} \in \overline{\tilde{G}}_E \setminus FSC(\tilde{G}_E)$, then repeating the same argument, show that in this case also $d(f_{e_1}, f_{e_2}, f_e) \leq \delta_{f_e}(\tilde{G}_E)$. Thus $\delta_{f_e}(\overline{\tilde{G}}_E) = sup\{d(f_{e_1}, f_{e_2}, f_e) : f_{e_1}, f_{e_2} \in \overline{\tilde{F}}_E\} \leq \delta_{f_e}(\tilde{G}_E)$. So $\delta_{f_e}(\tilde{G}_E) = \delta_{f_e}(\overline{\tilde{G}}_E)$. □

The converse of Theorem 4.1 is contained in the following theorem.

Theorem 4.3. *Let $(\tilde{F}_E, d)$ be a fuzzy soft 2-metric space. If $\tilde{F}_{e_n}$ is a sequence of fuzzy soft 2- closed sets with $\tilde{F}_{e_1} \supset \tilde{F}_{e_2} \cdots \supset \tilde{F}_{e_n} \supset \cdots$ such that $\delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0}$ as $n \to \infty\ \forall\ f_e \in \tilde{F}_E$, then $\bigcap_{n=1}^{\infty} \tilde{F}_{e_n}$ consists of a single point and $(\tilde{F}_E, d)$ is complete.*

Proof. Let $\{f_{e_n}\}$ be a Cauchy sequence of fuzzy soft points in $(\tilde{F}_E, d)$ and let $\tilde{F}_{e_n} = \{f_{e_n}, f_{e_{n+1}}, f_{e_{n+2}}, \cdots\}$. Then clearly, $\tilde{F}_{e_1} \supset \tilde{F}_{e_2} \cdots \supset \tilde{F}_{e_n} \supset \cdots$. Thus $\{\overline{\tilde{F}}_{e_n}\}$ is a decreasing sequence of fuzzy soft-2 closed sets. So for $f_e \in FSC(\tilde{F}_E)$ and arbitrary $\tilde{\tilde{\varepsilon}} > 0$, there is $n_1 \in N$ such that $d(f_{e_n}, f_{e_m}, f_e) < \tilde{\tilde{\varepsilon}}$ for all $n,\ m \geq n_1$. This show that $\delta_{f_e}(\tilde{F}_{e_{n_1}}) \leq \tilde{\tilde{\varepsilon}}$. By Theorem 3.15, $\delta_{f_e}(\overline{\tilde{F}}_{e_{n_1}}) \leq \tilde{\tilde{\varepsilon}}$. Since $\{\overline{\tilde{F}}_{e_{n_1}}\}$ is decreasing, $\delta_{f_e}(\tilde{F}_{e_n}) \leq \delta_{f_e}(\overline{\tilde{F}}_{e_{n_1}}) \leq \tilde{\tilde{\varepsilon}}$ for $n \geq n_1$. Hence $\delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0}$ as $n \to \infty$. By the given condition, $\bigcap_{n=1}^{\infty} \overline{\tilde{F}}_{e_n} = f_{e_2}$. This gives that $d(f_{e_n}, f_{e_2}, f_e) \leq \delta_{f_e}(\tilde{F}_{e_n}) \to \tilde{0}$ as $n \to \infty$ which implies $\{f_{e_n}\}$ converges to f_{e_2} in $(\tilde{F}_E, d)$ proving $(\tilde{F}_E, d)$ to be complete. □

Theorem 4.4. *A complete fuzzy soft 2-metric space $(\tilde{F}_E, d)$ satisfies the condition:* (A) *for any $f_e,\ f_{e'} \in \tilde{F}_E$, there exists a sequence $\{B_n\}$ of fuzzy soft closed 2-balls centered at $f_e,\ f_{e'}$ with $\delta_{f_{e_1}}(B_n) \to \tilde{0}$ as $n \to \infty\ \forall\ f_{e_1} \in \tilde{F}_E$ such that it cannot be written as a countable union of no-where dense fuzzy soft sets.*

Proof. Assume that, $\tilde{F}_E = \bigcup_{n \in N} \tilde{F}_{e_n} = \bigcup_{n \in N} \overline{\tilde{F}}_{e_n}$, where each $\tilde{F}_{e_n}$ is a no-where dense set and $\overline{\tilde{F}}_{e_n}$ does not contain any non-empty fuzzy soft 2-open set. Let $\tilde{G}_E$ be any fuzzy soft 2-open set. Since $\overline{\tilde{F}}_{e_1}$ is is no-where dense, $\overline{\tilde{F}}_{e_1}$ cannot contain $\tilde{G}_E$. Then there exists $\tilde{f}_{e_1} \in \tilde{G}_E$ such that $\tilde{f}_{e_1} \notin \overline{\tilde{F}_{e_1}}$. Since $\tilde{G}_E \setminus \overline{\tilde{F}_{e_1}}$ is fuzzy soft 2-open set and $\tilde{f}_{e_1} \in \tilde{G}_E \setminus \overline{\tilde{F}_{e_1}}$, by Lemma 3.9, there exist $f_{e_1'}, f_{e_2'}, \cdots, f_{e_n'}$ and $\tilde{\tilde{r}}_1, \tilde{\tilde{r}}_2, \cdots, \tilde{\tilde{r}}_n > \tilde{0}$ such that $\tilde{f}_{e_1} \in B_{\tilde{\tilde{r}}_1}(f_{e_1}, f_{e_1'}) \cap B_{\tilde{\tilde{r}}_2}(f_{e_1}, f_{e_2'}) \cap \cdots \cap B_{\tilde{\tilde{r}}_n}(f_{e_1}, f_{e_n'}) = \tilde{V}_E$ (say) $\subset \tilde{G}_E \setminus \tilde{F}_{e_1}$. Without loss of generality, because of the condition (A), choose $B_{\tilde{\tilde{r}}_1}(f_{e_1}, f_{e_1'})$ such that $\delta_{f_{e_1}}(B_{\tilde{\tilde{r}}_1}(f_{e_1} f_{e_1'})) < \tilde{1}\ \forall\ f_{e_1} \in FSC(\tilde{F}_E)$. Then $\delta_{f_{e_1}}(\tilde{V}_E) < \tilde{1}\ \forall\ f_{e_1} \in FSC(\tilde{F}_E)$. Choose $\tilde{S}_{E_1} = B_{\tilde{\tilde{r}}_1/2}(f_{e_1}, f_{e_1'}) \cap B_{\tilde{\tilde{r}}_2/2}(f_{e_1}, f_{e_2'}) \cap \cdots \cap B_{\tilde{\tilde{r}}_n/2}(f_{e_1}, f_{e_n'})$.

Then by Theorem 3.13, $\overline{\tilde{S}}_{E_1} \subset B_{\tilde{\tilde{r}}_1/2}[f_{e_1}, f_{e'_1}] \cap B_{\tilde{\tilde{r}}_2/2}[f_{e_1}, f_{e'_2}] \cap \cdots \cap B_{\tilde{\tilde{r}}_n/2}[f_{e_1}, f_{e'_n}] \subset \tilde{V}_E \subset \tilde{G}_E \setminus \tilde{F}_{e'}$ and $\delta_{f_{e_1}}(\tilde{S}_{E_1}) \leq \delta_{f_{e_1}}(\tilde{V}_E) < 1 \ \forall \ f_{e_1} \in FSC(\tilde{F}_E)$. Again, since $\tilde{S}_{E_1}$ is fuzzy soft 2-open set and $\tilde{F}_{e_2}$ is nowhere dense, $\tilde{S}_{E_1} \setminus \overline{\tilde{F}}_{e_2} \neq \phi$. Thus there exist $f_{e_2} \in \tilde{S}_{E_1} \setminus \overline{\tilde{F}}_{e_2}$. Proceeding as above another fuzzy soft 2-open set is $\tilde{S}_{E_2}$ such that $f_{e_2} \in \tilde{S}_{E_2} \subset \overline{\tilde{S}}_{E_2} \subset \tilde{S}_{E_1} \setminus \overline{\tilde{F}}_{e_2}$ and $\delta_{f_{e_1}}(\overline{\tilde{S}}_{E_2}) < \tilde{1}/2$ for all $f_{e_1} \in FSC(\tilde{F}_E)$. Continuing in this way, a sequence of fuzzy soft 2-closed sets are obtained in $\{\overline{\tilde{S}}_{E_n}\}$ such that $\overline{\tilde{S}}_{E_{n+1}} \subset \overline{\tilde{S}}_{E_n}$ for all $n \in N, \delta_{f_{e_1}}(\overline{\tilde{S}}_{E_n}) < \tilde{1}/n$ for all $f_{e_1} \in FSC(\tilde{F}_E)$. By Theorem 4.1, $\bigcap_{n=1}^{\infty} \tilde{S}_{E_n}$ is non empty and contains at most one point. Since $\{\overline{\tilde{S}}_{E_n}\} \cap \{\overline{\tilde{F}}_{e_n}\} = \phi$ for all $n \in N$ which is a contradiction. Hence the proof. □

5. CONCLUSION

2-metric spaces are interesting nonlinear generalizations of metric spaces which were conceived and studied in details by Gähler. Fuzzy soft set theory is a topic of interest in various fields due to its extensive potential for applications in numerous directions. Thus, intended to investigate a measurement among three different fuzzy soft points by considering the 2-metric space in fuzzy soft universe. As a result, introduced fuzzy soft 2-metric spaces and studied some of their topological properties. Also, extend the Cantor's Theorem and Baire Theorem to the fuzzy soft universe. This work can be extended to fixed-point theorems in fuzzy soft 2-metric spaces.

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References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[2] D. Molodtsov Soft set theory-first results, Comput. Math. Appl. 37(1999) 19–31.
[3] Ş. Güzide and N. Cağman, Soft closed sets on soft bitopological space, Journal of New Results in Science 3 (5) (2014) 57–66.
[4] Ş. Güzide and N. Cağman, Soft topological subspaces, Ann. Fuzzy Math. Inform. 10 (4) (2015) 525–535.
[5] Ş. Güzide, A new approach to Hausdorff space theory via the soft sets, Math. Probl. Eng. 9 (2016) 1–6.
[6] Güzide Şenel, Jeong-Gon Lee and Kul Hur, Distance and similarity measures for octahedron sets and their application to MCGDM problems, Mathematics, 8 (2020), 1690.
[7] Güzide Şenel, Jeong-Gon Lee and Kul Hur, Advanced soft relation and soft mapping, Int. J. Comput. Intell. Syst. 14 (1) (2021) 461–470.
[8] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[9] T. Beaula and C. Gunaseeli, On fuzzy soft metric spaces, Malaya J. Mat. 2 (2014) 197–202.
[10] A. F. Sayed and J. Ahmad, Fixed point theorems for fuzzy soft contractive mappings in fuzzy soft metric spaces, Ital. J. Pure Appl. Math. (2018) 200–214.
[11] A. Sayed, A. Alahmari and S. Omran, On fuzzy soft G-metric spaces, J. Adv. in Math. Comput. Sci. 27 (2018) 1–15.
[12] A. F. Sayed and A. Alahmari, Some new results of fixed point in fuzzy soft G-metric spaces, Eur. j. pure. appl. Math. 14 (2021) 923–941.
[13] S. Gähler, über die Uniformisierbarkeit 2-metrischer Räume, Math. Nachr. 28(1964) 235–244.

[14] Jeong-Gon Lee, Güzide Şenel, Jong-Il Baek, Sang Hyeon Han and Kul Hur, Neighborhood Structures and Continuities via Cubic Sets, Axioms, 11 (2022) 406.
[15] V. Çetkin, E. Güner and H. Aygün, On 2S-metric spaces, Soft Comput. 24 (2020) 12731–12742.
[16] P. K. Maji , R. K. Biswas and A. Roy, Fuzzy soft sets, J. Fuzzy Math. (2001) 589–602.
[17] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform 3 (2) (2012) 305–311.
[18] J. Mahanta and P. K. Das, Results on fuzzy soft topological spaces, (2012) arXiv preprint arXiv:1203.0634.
[19] S. Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. fuzzy Math. 20 (2012) 551–576.
[20] S. Das and S. Samanta, Soft metric, Ann. Fuzzy Math. Inform. 6 (1) (2013) 77–94.
[21] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Informa. 3 (2) (2012) 171–185.

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